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Situation 1: Investigating Properties of Operations

Prompt:

In a course for secondary teachers, students proved various properties of the real number operations of addition and multiplication starting from the field axioms of the real number system. For example, they proved such properties as "for all real numbers a and b, (-a)(-b)=ab. In an activity to assess their progress, students were asked to explore the properties of the following "funny addition" operation on the integers:

For all integers *a* and *b*, *a*☺*b* = $\begin{cases} a+b & if\ a\ is\ even \\ a-b & if\ a\ is\ odd \end{cases}$

Among other things, they were asked to compare the properties of funny addition with those of conventional integer addition and justify their conclusions. One student offered the following rationale that was similar to responses by several others in the class: "3☺2 = 1 and 2☺3 = 5. So the 3☺2 doesn't commute. To prove this in general, if x is even and y is odd then x☺y=x + y and y☺x = y-x . Since y-x ≠x+y, we can conclude that the funny addition is not commutative. "

Commentary:

Algebra is the study of operations and relations on sets of objects commonly represented by variables. In high school algebra the set of objects is usually the real numbers. Sometimes the set is restricted, as in the above example, to subsets of the real numbers. Teachers of high school algebra can build a better sense for operations and the interconnectedness of the operations' properties by reasoning about unusual operations. Many foci emerge in the above prompt. One mathematical focus is the understanding of what it means to say that an operation commutes on a set. Certainly this has branches reaching into areas of secondary mathematics such as matrix operations, function operations (e.g. composition), and geometric transformations to name the major ones. This understanding depends on students having a sense for the hidden quantifiers so often present but unstated in their reasoning. A second mathematical focus relates to proving or disproving that an operation has some given property. In the above case there seems to be a muddling of the logic of counterexamples and the norms for proving things in an algebraic context. A third focus is the broader mathematical question about where this operation came from, where it leads, what its properties actually are, and what larger problem space it represents.

Mathematical Focus 1

To reason about the properties of a binary operation, one must be clear about the set of numbers the operation is restricted to and the definition of the properties.

The commutative property of addition of real numbers states that for all *a* and *b* in the set of real numbers, a+b = b+a. This statement is typical of many such statements students encounter in algebra texts. In general, when a binary operation Ω is defined on a set S, the claim that Ω commutes on S means that for all *a* and *b* in S, $a \, \Omega \, b = b \, \Omega \, a$. To prove such a claim is false entails showing that there exists elements *a* and *b* in S such that $a \, \Omega \, b \neq b \, \Omega \, a$. The existence of even one pair of elements *a* and *b* in S such that $a \, \Omega \, b \neq b \, \Omega \, a$ is sufficient proof that Ω does not commute on S.

Whereas the commutative property involves only one quantifier, some common properties of operations are more complicated. For example, in the case of the addition of real numbers the "Existence of an additive Identity Element" asserts that there exists a real number *e* such that for all real numbers *a*, $e + a = a$ and $a + e = a$. The nested quantifiers in this statement make it more challenging for students to reason with, even though the notion of an identity element seems so simple. Likewise the "Existence of additive inverses" property of addition of real numbers asserts that for all real numbers a, there exists *a* number *b* such that $a+b = e$ and $b+a = e$, where *e* is the additive identity. Closure is another property that challenges student thinking. In addition to quantifiers, the property of closure for an operation requires students to focus on the set of numbers which the operation is restricted to when it is said to be "closed on the set."

In the case of the operation defined in the prompt, many students have difficulty proving that the operation has the "Existence of Identity" and the "Existence of Inverses" properties. In the case of closure they are often not sure where to begin except to demonstrate closure by using some specific examples or to say that the operation always makes sense or "always gives an answer." These difficulties are not unique to the situation in the prompt. Similar difficulties are encountered when students attempt to prove that multiplication of nxn matrices of real numbers does not have the "Existence of Inverses" property since they often have a problem negating the nested quantified statement that defines the property. In this regard, the composition of functions and the composition of transformations of geometric figures offer other good contexts for exploring the properties of operations.

Mathematical Focus 2

When students are unclear about the definition of a property of an operation, including the quantifiers in that definition, their reasoning about the operation is often incorrect and imprecise.

In the prompt, the student appears either to be unaware that a single counterexample falsifies the claim that ☺ is commutative or to be assuming some norms for proving that are unnecessary for the task at hand. In the context suggested in the prompt, the students had been using algebra to prove that addition and multiplication of real numbers had many familiar properties. They may not have been challenged to prove that some unfamiliar operation <u>did not</u> have some particular property. Confronted with such a challenge they might have felt compelled to imitate the kinds of proofs they had been doing and use a general algebraic argument. On the other hand, the students might have been unclear about the quantifier in the claim that an operation commutes. They may have felt that proving that the operation does not commute meant they had to prove that it did not commute most of the time or prove some claim about the circumstances under which it did not commute.

Sometimes students appear to mistake what is meant by the words *Identity* and *Inverse* and forget that they are always used relative to some specific operation. In the case of the ☺ operation, this is manifested in claims such as "☺ does not have the property of Inverses since the identity is 0 and 3☺$^{-}3 \neq 0$." It is not hard to construct operations on the real numbers that help to reveal such misunderstandings. For example, consider the operations where *a* and *b* are any real numbers: $a \oplus b = a + b + 1$ and $a \otimes b = ab + a + b$. These two operation on the real numbers have all of the properties of a field. However, -1 is the identity for $\oplus$ and 0 is the identity for $\otimes$. Even when students prove that -1 is the identity for the $\oplus$ operation, they often proceed to argue that $^{-}(a+1)$ is the inverse of *a* since $a \oplus {}^{-}(a+1) = a + {}^{-}(a+1) + 1 = 0$. Students seem to have difficulty giving up the notion that inverses have to combine to give you "0" in cases where the operation corresponds to the "addition" operation in the field structure and combine to give you "1" in cases where the operation corresponds to the "multiplication" of a field structure.

Mathematical Focus 3

Reasoning about operations and their properties gets at the very heart of algebra and is facilitated by the use of algebra.

The operation in the prompt is one example out of many similar operations one can have students explore. It is a non-commutative (non-abelian) group operation on the integers with identity element 0. To prove its properties one generally has to break the possibilities down into cases depending on whether the numbers are even or odd. For example, if *a* is even then its ☺-inverse is –*a*. If *a* is odd then its ☺-inverse is *a.* Here the teachers encounter an "addition" operation where the identity is still 0 but the negative integers are generally not the inverses of the positive integers relative to the operation. To prove ☺ is associative, one must show that for any integers a, b, and c, a☺(b☺c)= (a☺b)☺c. There are eight cases which are captured in the following table where "e" indicates the number is even and "o" indicates the number is odd. In each case the ☺ is eliminated from the expressions by using the even and odd characteristics of a, b, and c and the definition of the operation.

a	**b**	**c**	**a☺(b☺c)**	**(a☺b)☺c**
e	e	e	*a+(b+c)*	*(a+b)+c*
e	e	o	*a+(b+c)*	*(a+b)+c*
e	o	e	*a+ (b-c)*	*(a+b)-c*
e	o	o	*a+ (b-c)*	*(a+b)-c*
o	e	e	*a- (b+c)*	*(a-b)-c*
o	e	o	*a- (b+c)*	*(a-b)-c*
o	o	e	*a-(b-c)*	*(a-b)+c*
o	o	o	*a-(b-c)*	*(a-b)+c*

After making the table students conclude that in each case the operation is associative and notice that the value of c did not matter in the comparison of a☺(b☺c) with (a☺b)☺c. Hence, only four cases needed to be considered.

The operation is a generalization of a similar operation applied to the integers modulo n, where n is an integer greater than 1:

For all *a* and *b* elements of Z_n (the integers modulo n),

$$a☺b = \begin{cases} (a+b) \ mod \ n & if \ a \ is \ even \\ (a-b) mod \ n & if \ a \ is \ odd \end{cases}$$

When n is odd, the operation on Z_n is not associative and fails in other important aspects of group operations. When n is even, the operation is a group operation on Z_n and is algebraically the same as the group of symmetries of a regular polygon with n/2 sides.

There are many contexts using matrix multiplication or function composition that lead student to similar investigations of the properties of matrix and function operations. Matrix multiplication is not commutative (nor is function composition) unless significant restrictions are placed on the set of objects the operations are applied to. Secondary mathematics teachers can be challenged to create restricted systems in which matrix multiplication does commute on the restricted set of matrices.